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On an Extremal Problem in the Theory of Rational Approximation

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1. INTRODUCTION

Let G be an n-connected domain in $\mathbb{C}$ bounded by simple analytic curves $\gamma_1,\ldots,\gamma_n$. Let $\Gamma=\bigcup_{j=1}^n \gamma_j$. Let $R(G)$ be the uniform closure in $\bar{G}$ of the algebra of rational functions with poles outside of $\bar{G}$. We recall that, for domains with analytic boundary, one can define a Smirnov class $E_1(G)$ as the collection of analytic functions in G whose boundary values belong to the closure of $R(G)$ in $L^1(ds)$. Here, ds is Lebesgue measure on Γ. (Details concerning the general definition and properties of the classes E_p can be found in [3, 5, 9].)

In [6] the following concept of *rational capacity* λ has been introduced:

$$\lambda=\lambda(G) \stackrel{\text{def}}{=} \inf_{\phi\in R(G)} \|\bar{\zeta}-\phi(\zeta)\|.$$

Here, $\|\cdot\|$ denotes the uniform norm on $\bar{G}$. The importance of λ can be seen from a simple observation that $\lambda=0$ if and only if G degenerates to the union of analytic arcs. Also, it turns out that λ enjoys simple estimates in terms of elementary geometric characteristics of G: area and perimeter, i.e.,

$$\sqrt{\frac{a(G)}{\pi}} \geqslant \lambda \geqslant \frac{2a(G)}{P(G)}. \tag{$*$}$$

Here, $a(G)=a$ and $P(G)=P$ denote the area and perimeter $(=\int_\Gamma ds)$ of G, respectively. The first inequality in $(*)$ was proved in [1] and the second in [6].

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Using a standard duality argument (see [3], for the case of the unit disk [4–8]) one can easily show that

$$\lambda(G) \overset{\text{def}}{=} \inf_{\phi \in R(G)} \|\bar{\zeta} - \phi(\zeta)\| = \inf_{\phi \in R(G)} \|\bar{\zeta} - \phi(\zeta)\|_{\Gamma}$$

$$= \sup_{\substack{f \in E_1(G) \\ \|f\|_{E_1} \leq 1}} \left\{ \left| \int_{\Gamma} f(\zeta)\, \bar{\zeta}\, d\zeta \right| \right\}. \tag{1}$$

Also, it is not hard to see (via F. and M. Riesz and Banach–Alaoglu theorems) that there exist $\phi^* \in H^\infty(G)$ and $f^* \in E_1(G)$, the extremal functions for which the infimum and supremum are attained. Moreover, as we assume Γ to be analytic, i.e., $\bar{\zeta}|_{\Gamma} = S(\zeta)$, where $S(\zeta)$ is the so-called Schwarz function analytic in a tubular neighborhood of Γ (see [2]), it follows from a result of S. Ya. Khavinson (see [7, 8]) that both functions f^* and ϕ^* can be analytically continued across Γ.

In this paper in Section 2 we obtain a simple expression for the area of the image $\phi^*(G)$ in terms of λ and the number of zeros N_{f^*} of the extremal funtion $f^*(z)$ (zeros on Γ are counted with a half-multiplicity).

In Section 3 as a corollary of this theorem we obtain the classical isoperimetric inequality with sharp constants (Corollary 3). Also, we show (Corollary 2) that the area of the image of the best approximation ϕ^* to $\bar{\zeta}$ in simply connected domains is always dominated by the "isoperimetric deficiency" $1 - 4\pi a/P^2$ of G.

In particular this estimate gives another proof of the well-known fact that equality in the isoperimetric inequality occurs if and only if the domain is a disk.

2. The Main Theorem

We keep the same notation as above.

THEOREM. *Let ϕ^* and f^* be the extremal functions in the problem* (1). *Let N_{f^*} denote the number of zeros of f^* in G. (Zeros on Γ are counted with half-multiplicity.) Then*

$$\iint_G |(\phi^*)'|^2 dx\, dy = (\textit{area of } \phi^*(G) \textit{ with multiplicity})$$
$$= -a + \lambda \operatorname{Im} \int_{\Gamma} \frac{\overline{f^*}}{|f^*|}\, ds - \pi(2 - n + N_{f^*})\, \lambda^2. \tag{2}$$

Note. By continuity, we extend the function $\overline{f^*}/|f^*| = e^{-i\arg f^*}$ to the points where f^* vanishes.

Proof. The following little argument is due to S. Ya. Khavinson (see [7, 8]). For any $\phi \in R(G)$, $f \in E_1(G)$, $\|f\|_{E_1} \leqslant 1$, we have

$$\begin{aligned} \|\bar{\zeta} - \phi\|_\Gamma &\geqslant \int_\Gamma |\bar{\zeta} - \phi(\zeta)|\, |f(\zeta)|\, ds \\ &\geqslant \left| \int_\Gamma (\bar{\zeta} - \phi(\zeta))\, f(\zeta)\, d\zeta \right| \\ &= \left| \int_\Gamma \bar{\zeta} f(\zeta)\, d\zeta \right| \leqslant \lambda. \end{aligned} \tag{3}$$

If f^*, ϕ^* are extremal and if we assume without loss of generality that

$$\lambda = \int \bar{\zeta} f^*(\zeta)\, d\zeta,$$

then everywhere in (3) equalities hold. Therefore,

$$[\bar{\zeta} - \phi^*(\zeta)]\, f^*(\zeta)\, d\zeta = \lambda |f^*(\zeta)|\, ds \tag{4}$$

a.e. on Γ. Since Γ is analytic and ϕ^*, f^* are analytic near Γ, we conclude that (4) holds everywhere on Γ. Let us rewrite (4) in the form

$$\phi^*(\zeta) = \bar{\zeta} - \lambda \frac{\overline{f^*(\zeta)}}{|f^*(\zeta)|} \frac{d\bar{\zeta}}{ds}. \tag{5}$$

Set

$$\tau(\zeta) = \frac{\overline{f^*(\zeta)}}{|f^*(\zeta)|}$$

and extend it by continuity to the points where f^* vanishes. From (5), it follows that on Γ we have

$$d\phi^* = d\bar{\zeta} - \lambda\, d\tau \frac{d\bar{\zeta}}{ds} - \lambda \tau d\left(\frac{d\bar{\zeta}}{ds}\right). \tag{6}$$

According to Stokes' theorem

$$\int\int_G |(\phi^*)'|^2 dx\, dy = \frac{1}{2i} \int_\Gamma \overline{\phi^*}\ \ d\phi^*.$$

Then, using (5), (6) and taking into account that on Γ $|\tau| = |d\zeta/ds| = 1$, we obtain

$$\iint_G |(\phi^*)'|^2 dx\, dy = \frac{1}{2i}\left\{\int_\Gamma \zeta\, d\bar{\zeta} - \lambda \int_\Gamma \zeta \frac{d\bar{\zeta}}{ds}\, d\tau - \lambda \int_\Gamma \zeta\tau d\left(\frac{d\bar{\zeta}}{ds}\right) - \lambda \int_\Gamma \bar{\tau}\frac{d\zeta}{ds}\, d\bar{\zeta} + \lambda^2 \int_\Gamma \bar{\tau}\, d\tau + \lambda^2 \int_\Gamma \frac{d\zeta}{ds}\, d\left(\frac{d\bar{\zeta}}{ds}\right)\right\}. \tag{7}$$

Applying Stokes' theorem again, we obtain

$$\int_\Gamma \zeta\, d\bar{\zeta} = -2ia. \tag{8}$$

Also,

$$\int_\Gamma \frac{d\zeta}{ds}\, d\left(\frac{d\bar{\zeta}}{ds}\right) = \int_\Gamma \frac{d(d\bar{\zeta}/ds)}{d\bar{\zeta}/ds} = i\Delta_\Gamma \arg\left(\frac{d\bar{\zeta}}{ds}\right) = -2\pi i(2-n). \tag{9}$$

Integration by parts gives us

$$-\lambda \int_\Gamma \zeta \frac{d\bar{\zeta}}{ds}\, d\tau = \lambda \int_\Gamma \tau ds + \lambda \int_\Gamma \tau\zeta d\left(\frac{d\bar{\zeta}}{ds}\right).$$

So

$$-\lambda \int_\Gamma \zeta \frac{d\bar{\zeta}}{ds}\, d\tau - \lambda \int_\Gamma \zeta\tau d\left(\frac{d\bar{\zeta}}{ds}\right) - \lambda \int_\Gamma \bar{\tau}\frac{d\zeta}{ds}\, d\bar{\zeta} = \lambda \int_\Gamma \tau ds - \lambda \int_\Gamma \bar{\tau} ds = 2i \operatorname{Im} \int_\Gamma \tau ds. \tag{10}$$

Finally, since $|\tau| = 1$ on Γ,

$$\int_\Gamma \bar{\tau}\, d\tau = \int_\Gamma \frac{d\tau}{\tau} = i\Delta_\Gamma \arg \tau = -i\Delta_\Gamma \arg f^* = -2\pi i N_{f^*}. \tag{11}$$

Combining formulas (7)–(11) we obtain (2).

3. Applications

We keep the same notations as in Sections 1 and 2.

COROLLARY 1.

$$\int_G\int |(\phi^*)'|^2 dx\, dy = (\textit{area of } \phi^*(G) \textit{ with multiplicity})$$
$$\leqslant \lambda P - a - \lambda^2\pi(2 - n + N_{f^*}) \tag{12}$$

Proof. Since

$$\left|\int_\Gamma \frac{\overline{f^*}}{|f^*|}\, ds\right| \leqslant P$$

(12) immediately follows from (2).

COROLLARY 2. *If G is simply connected, then*

$$\int\int_G |(\phi^*)'|^2 dx\, dy \leqslant \frac{P^2}{4\pi}\left(1 - \frac{4\pi a}{P^2}\right). \tag{13}$$

Proof. As $n = 1$, $N_{f^*} \geqslant 0$ and since $q(\lambda) = \lambda P - a - \pi\lambda^2$ attains its maximum at $\lambda = P/2\pi$, (13) follows directly from (12).

COROLLARY 3. (Isoperimetric inequality).

$$4\pi a \leqslant P^2 \tag{14}$$

Moreover, equality occurs in (14) *if and only if G is a disk of radius* λ.

Proof. For $n = 1$ (14) follows from (13). If $n > 1$, then (14) holds for the interior G_n of γ_n (we assume that γ_n is an outer contour). But $a = a(G) < a(G_n)$ and $P(G) \geqslant P(G_n)$. So, (14) is verified for $n > 1$.

If equality occurs in (14), then it is clear that $n = 1$. Then, according to (13) $4\pi a = P^2$ if and only if $(\phi^*)' \equiv 0$, i.e., $\phi^* \equiv \text{const}$. Therefore, $|\zeta - \text{const}|\,|_\Gamma \equiv \lambda$. Thus, G is a disk of radius λ.

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